Proof Proposition

of the Syracuse Conjecture

by Rémy Aumeunier

The Syracuse conjecture, known as the Collatz problem, was introduced by the German mathematician Lothar Collatz in 1937. It is the unproven mathematical statement that, starting from any integer $n > 0$, the sequence U_n arrives at the value 1, after following these rules :

$$U_0 \in \mathbb{N}^* \quad U_{n+1} = \begin{cases} \frac{U_n}{2} & \text{if } U_n \text{ is even} \\ 3U_n + 1 & \text{if } U_n \text{ is odd} \end{cases}$$

1 Preamble

This proposed method involves transforming the sequence U_n into a polynomial form. The analysis of the polynomialization of the Syracuse sequence offers an unconventional approach to thoroughly examine the behavior of these sequences. Through this perspective, I aim to provide insights that could contribute to answering the conjecture or approaching this problem in an unexpected way.

2 Polynomialization of the Syracuse Sequence

To perform the polynomialization of the elements of the Syracuse sequence, I use a variant of the Horner method known as Ruffini-Horner. This method associates a value with a polynomial representation.

$$U_0 \in \mathbb{N}^* \quad U_{n+1} = \begin{cases} \frac{U_n}{2} & \text{if } U_n \text{ is even} \\ 3U_n + 1 & \text{if } U_n \text{ is odd} \end{cases}$$

$$\{q_x, p_x\} \in \mathbb{N} \quad u_0 \frac{3^p}{2^{q_0}} + \frac{3^{p-1}}{2^{q_1}} + \frac{3^{p-2}}{2^{q_2}} + \frac{3^{p-3}}{2^{q_3}} + \frac{3^{p-4}}{2^{q_4}} + \cdots + \frac{3^{p_0}}{2^{q_n}} = u_n$$

And if I consider the compressed Syracuse sequence

$$U_0 \in \mathbb{N}^* \quad U_{n+1} = \begin{cases} \frac{U_n}{2} & \text{if } U_n \text{ is even} \\ \frac{3U_n+1}{2} & \text{if } U_n \text{ is odd} \end{cases}$$

After arrangement, I obtain a form that I will qualify as canonical. Since a multiplication by 3 necessarily involves a division by 2.

$$u_0 \cdot \left(\frac{3}{2}\right)^p \cdot \frac{1}{2^{q_0}} + \left(\frac{3}{2}\right)^{p-1} \cdot \frac{1}{2^{q_1}} + \left(\frac{3}{2}\right)^{p-2} \cdot \frac{1}{2^{q_2}} + \left(\frac{3}{2}\right)^{p-3} \cdot \frac{1}{2^{q_3}} + ... + \left(\frac{3}{2}\right)^{p-n} \cdot \frac{1}{2^{q_p}} = u_n$$

Thus, each value of the sequence can be associated with a distinct transposition. This transposition follows a common structure systematically imposed by the Syracuse method. Here, p denotes the number of multiplications by 3, while the highest power of 2 reflects the number of divisions by 2 performed.

2.1 Numerical Application

The polynomialization of the Syracuse sequence is a simple transposition of values

$$U_0 \in \mathbb{N}^* \quad U_{n+1} = \begin{cases} \frac{U_n}{2} & \text{si } U_n \text{ est un entier pair} \\ 3U_n + 1 & \text{si } U_n \text{ est un entier impair} \end{cases}$$

$$U_{15} = \{46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1\}$$

I write all intermediate values as fractions.

$$\frac{\left(\frac{\left(\frac{\left(\frac{\left(\frac{15\cdot 3+1}{2}\right)\cdot 3+1}{2}\right)\cdot 3+1}{2}\right)\cdot 3+1}{2\cdot 2\cdot 2\cdot 2\cdot 2}\right)\cdot 3+1}{2\cdot 2\cdot 2\cdot 2}$$

Which I then rearrange as a pseudo-polynomial.

$$\left(...\right) + \frac{1}{2^4}$$

$$\left(...\right) + \frac{3}{2^9} + \frac{1}{2^4}$$

$$\left(...\right) + \frac{3^2}{2^{10}} + \frac{3}{2^9} + \frac{1}{2^4}$$

$$\cdots$$

$$15\frac{3^5}{2^{12}} + \frac{3^4}{2^{12}} + \frac{3^3}{2^{11}} + \frac{3^2}{2^{10}} + \frac{3^1}{2^9} + \frac{3^0}{2^4}$$

3 Proof Proposal

In this proof proposal, I will consider the intermediate values from their representation as a pseudo-polynomial. Then, I will study the behavior of q.

$$u_0 \cdot \left(\frac{3}{2}\right)^p \cdot \frac{1}{2^{q_0}} + \left(\frac{3}{2}\right)^{p-1} \cdot \frac{1}{2^{q_1}} + \left(\frac{3}{2}\right)^{p-2} \cdot \frac{1}{2^{q_2}} + ... + \left(\frac{3}{2}\right)^{p-n} \cdot \frac{1}{2^{q_p}} = u_n$$

u_n represents the nth value of the Syracuse sequence, with u_0 being the first element of the sequence. The variable p counts the odd values encountered during the iterations, while q_0 corresponds to the number of supernumerary divisions by 2 related to the even integers of the form $(2^{p>1}n)$. It is precisely the evolution of q_0, which increases monotonically because q_0 and the sum of the supernumerary divisions by 2, that demonstrates the inevitable convergence of the sequence to 1. Indeed, the limit of q_0 allows us to establish an order relation. And having $u_0 \cdot \frac{3^p}{2^{p+q_0}} \approx 0$ makes the sequence converge to 1.

$$\lfloor \frac{u_0 \cdot 3^p}{2^{p+q_0}} \rfloor + \lfloor \frac{3^{p-1} + \cdots + 3^{p_0} \cdot 2^{b_n}}{2^{p+q_0}} \rfloor + (u_0 \cdot 3^p + 3^{p-1} + \ldots 2^{b_n}) mod(2^{p+q_0})$$

$$0 + 0 + \frac{u_0 \cdot 3^p + 3^{p-1} + \cdots + 3^{p_0} \cdot 2^{b_n}}{2^{p+q_0}} = \frac{2^{p+q_0}}{2^{p+q_0}} = 1$$

3.1 Numerical Application :

With the divisions I have classified as extra in bold.

$u_n = \{23 - 70 - 35 - 106 - 53 - 160 -$ **80 - 40 - 20 - 10** $- 5 - 16 -$ **8 - 4 - 2** $- 1\}$

$$\frac{23 \cdot 3 + 1}{2} = 35$$

$$34.5 + 0.5 = 35 \quad \frac{3}{2} = 1.5$$

$$\frac{23 \cdot 3^2}{2^2} + \frac{3}{2^2} + \frac{1}{2} = 53$$

$$51.75 + 0.75 + 0.5 = 53 \quad \left(\frac{3}{2}\right)^2 = 2.25$$

$$\left(\frac{23 \cdot 3^3}{2^3} + \frac{3^2}{2^3} + \frac{3}{2^2} + \frac{1}{2}\right) \div 2^4 = 5$$

$$\frac{23 \cdot 3^3}{2^3} \cdot \frac{1}{2^4} + \frac{3^2}{2^3} \cdot \frac{1}{2^4} + \frac{3}{2^2} \cdot \frac{1}{2^4} + \frac{1}{2} \cdot \frac{1}{2^4} = 5$$

$$\left(\frac{3}{2}\right)^3 \cdot \frac{1}{2^4} = 0.2109375$$

$$4.8515625 + 0.0703125 + 0.046875 + 0.03125 = 5$$

$$\left(\frac{23\cdot 3^4}{2^4}\cdot\frac{1}{2^4}+\frac{3^3}{2^3}\cdot\frac{1}{2^5}+\frac{3^2}{2^2}\cdot\frac{1}{2^5}+\frac{3}{2}\cdot\frac{1}{2^5}+\frac{1}{2}\right)\div 2^3=1$$

$$\left(\frac{23\cdot 3^4}{2^4}\cdot\frac{1}{2^7}+\frac{3^3}{2^3}\cdot\frac{1}{2^8}+\frac{3^2}{2^2}\cdot\frac{1}{2^8}+\frac{3}{2}\cdot\frac{1}{2^8}+\frac{1}{2^4}\right)=1$$

$$\left(\frac{3}{2}\right)^4\cdot\frac{1}{2^7}=0.03955078125$$

$$0.90966796875+0.01318359375+0.0087890625+0.068359375=1$$

Here, the coefficient $\frac{1}{2^{q_0}}=\frac{1}{2^7}$ because there are 7 extra divisions, which are associated with the values **80 - 40 - 20 - 10 - 8 - 4 - 2**.

3.2 Frequency $(2^{n>1}\cdot n)$

In the academic corpus, there are numerous proofs establishing the impossibility of an infinite alternation of even and odd integers. What follows is not an academic proof, but it serves to justify the presence of integers of the form $(2^{p>1}n)$. For this, I consider the Syracuse sequence associated with the integer 31.

$$\{31,94,47,142,71,214,107,322,161,484,242,121\}$$

Then, the compressed form of Syracuse and the even integers, and I calculate the multiplier coefficient.

$$\{94=2\cdot 47,142=2\cdot 71,214=2\cdot 101,322=2\cdot 7\cdot 23,484=2^2\cdot 11,...\}$$

$$142/94=1.5106...\quad,\quad 214/142=1.5070...\quad,\quad 322/214=1.5045...$$

This coefficient, not being an integer, does not allow for the reuse of the large prime numbers present in the factorization of the elements of the Syracuse sequence. This implies that the powers appear very quickly on small prime numbers. And if I consider a pair (odd/even), these two integers are coprime "$(n_x\cdot 3+1)$", making the integers of the form $(2^{p>1}n)$ obligatory and frequent in a relatively close neighborhood. Specifically, this amounts to trying to multiply by $\approx 3=2\cdot(1.5\cdots)$ without being able to use this prime number in its decomposition, which imposes a fairly significant frequency of integers of the form $2^{n>1}\cdot n$ in the sequence, which makes it converge, because $u_0\cdot\frac{3^p}{2^{p+q_0}}\approx 0.\cdots$.

3.3 Unicity of the Cycle

In this proposed demonstration of the uniqueness of the cycle, I will consider two occurrences of the same value

$$\frac{u_0\cdot 3^p+3^{p-1}+\cdots+2^{b_n}}{2^{p+q_0}}=\frac{3n_a+2^{b_n}}{2^{p+q_0}}$$

$$u_m = \frac{3n_a + 2^a}{2^c} = \frac{3n_b + 2^b}{2^d} = u_n$$

$$\frac{(3n_a + 2^a)\cdot 2^d}{2^c 2^d} = \frac{(3n_b + 2^b)\cdot 2^c}{2^c 2^d} \quad , \quad 2^a < 2^c < 2^b < 2^d$$

$$\frac{(3n_a + 2^a)\cdot 2^{d-c}\cdot 2^c}{2^c 2^d} = \frac{(3n_b + 2^b)\cdot 2^c}{2^c 2^d}$$

$$u_n = \frac{(3n_a + 2^a)\cdot 2^{d-c}}{2^d} = \frac{(3n_b + 2^b)}{2^d} \neq 1 \in \mathbb{N}^*$$

The only possible integer values for u_n will be of the form 2^n, which demonstrates the uniqueness of the cycle $\{4, 2, 1, 3, 4, 2, 1, \ldots\}$

3.4 Generalization by Substitution

If desired, we can generalize the calculation of the sequence here, replacing 3 with 5.

$$U_0 \in \mathbb{R}^* \quad U_{n+1} = \begin{cases} \frac{U_n}{2} & \text{si } U_n \text{ est un entier pair} \\ 5U_n + 1 & \text{si } U_n \text{ est un entier impair} \end{cases}$$

For $3x + 1$, I have :

$$\frac{3^n}{2^{(n+\frac{2n}{3})}} < 1$$

Whereas, for the $5x + 1$ case, I have :

$$\frac{5^n}{2^{(2n+\frac{n}{3})}} < 1$$

This implies that only 2/3 of the divisions by 2 are needed in addition to those related to the canonical form, whereas for the sequence $5x + 1$, more than twice as many are required. This causes the sequence to diverge, except for a few rare or very specific cases, if they exist.

3.5 Recurrence

This approach also allows reasoning by recurrence. If I consider a large integer u_0, each time the integer division part in the calculation of u_n equals zero, I can start over from this integer (which is different from 1) and no longer consider the old transposition, to build a new sequence, and so on until the sequence equals 1.

$$0 + n_x \cdot \frac{2^{q_0}}{2^{q_0}} = u'_0$$

$$u'_0 \frac{3^p}{2^{q_0}} + \frac{3^{p-1}}{2^{q_0}} + \frac{3^{p-2}}{2^{q_1}} + \frac{3^{p-3}}{2^{q_2}} + \frac{3^{p-4}}{2^{q_3}} + \cdots + \frac{3^{p_0}}{2^{q_n}} = 1$$

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4 Acknowledgement

This proof proposition wouldn't have been as straightforward and therefore hardly contestable without ChatGPT ; this AI enabled me to explore some dead ends and approach problems in a way that excludes unnecessary complexities.This was achieved despite its occasional quirks, to remain polite."

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