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New upper bounds for the numerical radius of Hilbert space operators ☆

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A B S T R A C T

In this paper we present new upper bounds for the numerical radius of bounded linear operators defined on a complex Hilbert space. Further we obtain estimations for upper bounds for the numerical radius of the sum of the product of bounded linear operators. We show that the bounds obtained here improve on the existing well-known upper bounds.

1. Introduction

Let $\mathbb{H}$ be a complex Hilbert space with inner product $\langle .,. \rangle$ and $\mathcal{B}(\mathbb{H})$ be the collection of all bounded linear operators on $\mathbb{H}$. As usual the norm induced by the inner product $\langle .,. \rangle$ is denoted by $\|.\|$. For $A \in \mathcal{B}(\mathcal{H})$, let $\|A\|$ be the operator norm of A, i.e., $\|A\| =$

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$\sup_{\|x\|=1} \|Ax\|$. For $A \in \mathcal{B}(\mathcal{H})$, A^* denotes the adjoint of A and $|A|, |A^*|$ respectively denote the positive part of A, A^*, i.e., $|A| = (A^*A)^{\frac{1}{2}}, |A^*| = (AA^*)^{\frac{1}{2}}$. Let $S_{\mathbb{H}}$ denote the unit sphere of the Hilbert space $\mathbb{H}$. The numerical range of A, denoted by $W(A)$, is defined as

$$W(A) := \left\{ \langle Ax, x \rangle \quad : \quad x \in \mathcal{H}, \quad \|x\| = 1 \right\}.$$

Considering the continuous mapping $x \longmapsto \langle Ax, x \rangle$ from $S_{\mathbb{H}}$ to the scalar field $\mathbb{C}$, it is easy to see that $W(A)$ is a compact subset of $\mathbb{C}$ if $\mathbb{H}$ is finite dimensional.

The numerical radius of A, denoted as $w(A)$, is defined as

$$w(A) := \sup_{\|x\|=1} |\langle Ax, x \rangle|.$$

The numerical radius is a norm on $\mathcal{B}(\mathbb{H})$ satisfying the following inequality

$$\frac{\|A\|}{2} \leq w(A) \leq \|A\|. \tag{1.1}$$

Clearly, (1.1) implies that the numerical radius norm is equivalent to the operator norm. The inequality (1.1) is sharp, $w(A) = \|A\|$ if $AA^* = A^*A$ and $w(A) = \frac{\|A\|}{2}$ if $A^2 = 0$. Kittaneh [8, Th. 1] and [9, Th. 1] improved on the inequality (1.1), to prove that

$$\frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\| \leq w^2(A) \leq \frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\|. \tag{1.2}$$

$$w(A) \leq \frac{1}{2}\|A\| + \frac{1}{2}\|A^2\|^{\frac{1}{2}}. \tag{1.3}$$

In [4, Cor. 2.5] authors improved on the right hand inequalities of both (1.1) and (1.2) to prove that

$$w^2(A) \leq \min_{0 \leq \alpha \leq 1} \left\| \alpha |A|^2 + (1-\alpha)|A^*|^2 \right\|. \tag{1.4}$$

The inequality (1.1) has been studied and improved by many mathematicians, we refer to [1–3,5] and references therein.

In this paper we first obtain an upper bound for the numerical radius of bounded linear operators on $\mathbb{H}$, we prove that if $A \in \mathcal{B}(\mathbb{H})$, then

$$w^2(A) \leq \frac{1}{4} \left\| |A|^2 + |A^*|^2 \right\| + \frac{1}{2} w\left(|A||A^*|\right).$$

This inequality is stronger than the right hand inequality in (1.2) as well as the inequality (1.3). We next obtain upper bounds for the numerical radius of the sum of the product of bounded linear operators. In particular, we show that, if $A, B \in \mathcal{B}(\mathbb{H})$ and $r \geq 2$, then

$$w^r(A^*B) \leq \frac{1}{2} w^2 \left(|B|^r + \mathrm{i}|A|^r\right) \leq \frac{1}{2} \left\| |B|^{2r} + |A|^{2r} \right\|.$$

This is an improvement of the Dragomir's inequality [7], namely, $w^r(A^*B) \leq \frac{1}{2} \left\| |B|^{2r} + |A|^{2r} \right\|$.

2. Main results

We begin this section with the following sequence of lemmas which will be used to reach our goal in this present article. First lemma is known as a generalized mixed Cauchy-Schwarz inequality which involves two nonnegative continuous functions.

Lemma 2.1. *([10, Th. 5]). Let $A \in \mathcal{B}(\mathbb{H})$. Let f and g be nonnegative functions on $[0, \infty]$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty]$. Then*

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\| \|g(|A^*|)y\|,$$

for all $x, y \in \mathbb{H}$.

Second lemma deals with positive operators.

Lemma 2.2. *([12, p. 20]). Let $A \in \mathcal{B}(\mathbb{H})$ be positive, i.e., $A \geq 0$. Then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle,$$

for all $r \geq 1$ and for all $x \in \mathbb{H}$ with $\|x\| = 1$.

Third lemma is known as Buzano's inequality.

Lemma 2.3. *([6]) Let $x, y, e \in \mathcal{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle\langle e, y\rangle| \leq \frac{1}{2} \left(\|x\|\|y\| + |\langle x, y\rangle|\right).$$

Fourth lemma is known as Bohr's inequality which deals with positive numbers.

Lemma 2.4. *([13]) For $i = 1, 2, \ldots, n$, let a_i be a positive real number. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1} \sum_{i=1}^{n} a_i^r,$$

for all $r \geq 1$.

We now present the first inequality.

Theorem 2.5. *Let $A \in \mathcal{B}(\mathbb{H})$. Then*

$$w^{2r}(A) \leq \frac{1}{4}\left\||A|^{2r} + |A^*|^{2r}\right\| + \frac{1}{2}w\left(|A|^r|A^*|^r\right),$$

for all $r \geq 1$.

Proof. Let $x \in \mathbb{H}$ with $\|x\| = 1$. Considering $f(t) = g(t) = t^{\frac{1}{2}}$ in Lemma 2.1 we have that

$$|\langle Ax, x\rangle|^2 \leq \langle |A|x, x\rangle\langle |A^*|x, x\rangle.$$

It follows from Lemma 2.2 that

$$|\langle Ax, x\rangle|^{2r} \leq \langle |A|^r x, x\rangle\langle |A^*|^r x, x\rangle = \langle |A^*|^r x, x\rangle\langle x, |A|^r x\rangle.$$

From Lemma 2.3 we have,

$$\langle |A^*|^r x, x\rangle\langle x, |A|^r x\rangle \leq \frac{1}{2}\left\||A|^r x\right\| \left\||A^*|^r x\right\| + \frac{1}{2}\left|\langle |A^*|^r x, |A|^r x\rangle\right|.$$

$$\begin{aligned}
\text{So we get, } |\langle Ax, x\rangle|^{2r} &\leq \frac{1}{4}\left(\left\||A|^r x\right\|^2 + \left\||A^*|^r x\right\|^2\right) + \frac{1}{2}\left|\langle |A|^r|A^*|^r x, x\rangle\right| \\
&= \frac{1}{4}\left(\langle |A|^{2r}x, x\rangle + \langle |A^*|^{2r}x, x\rangle\right) + \frac{1}{2}\left|\langle |A|^r|A^*|^r x, x\rangle\right| \\
&= \frac{1}{4}\left\langle\left(|A|^{2r} + |A^*|^{2r}\right)x, x\right\rangle + \frac{1}{2}\left|\langle |A|^r|A^*|^r x, x\rangle\right| \\
&\leq \frac{1}{4}\left\||A|^{2r} + |A^*|^{2r}\right\| + \frac{1}{2}w\left(|A|^r|A^*|^r\right).
\end{aligned}$$

Therefore, taking supremum over $\|x\| = 1$ we get,

$$w^{2r}(A) \leq \frac{1}{4}\left\||A|^{2r} + |A^*|^{2r}\right\| + \frac{1}{2}w\left(|A|^r|A^*|^r\right),$$

as required. □

The following corollary is an immediate consequence of Theorem 2.5.

Corollary 2.6. *Let $A \in \mathcal{B}(\mathbb{H})$. Then*

$$w^2(A) \leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| + \frac{1}{2}w\left(|A||A^*|\right).$$

Remark 2.7. 1. If $|A||A^*| = 0$, then it follows from Corollary 2.6 and the left hand inequality of (1.2) that $w^2(A) = \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|$.

2. The inequality in Corollary 2.6 improves on the right hand inequality in (1.2). Clearly, $w\left(|A||A^*|\right) \leq \||A||A^*|\| = \left\|A^2\right\|$. Therefore,

$$\begin{aligned}
w^2(A) &\leq \frac{1}{4}\left\||A|^2+|A^*|^2\right\|+\frac{1}{2}w\left(|A||A^*|\right)\\
&\leq \frac{1}{4}\left\||A|^2+|A^*|^2\right\|+\frac{1}{2}\left\|A^2\right\|\\
&\leq \frac{1}{4}\left\||A|^2+|A^*|^2\right\|+\frac{1}{4}\left\||A|^2+|A^*|^2\right\|\\
&= \frac{1}{2}\left\||A|^2+|A^*|^2\right\|.
\end{aligned}$$

Thus, the inequality Corollary 2.6 improves on the right hand inequality in (1.2).

3. The inequality in Corollary 2.6 improves on the inequality in (1.3).

$$\begin{aligned}
\text{Clearly,}\quad w^2(A) &\leq \frac{1}{4}\left\||A|^2+|A^*|^2\right\|+\frac{1}{2}w\left(|A||A^*|\right)\\
&\leq \frac{1}{4}\left\||A|^2+|A^*|^2\right\|+\frac{1}{2}\left\|A^2\right\|\\
&\leq \frac{1}{4}\left\|A^2\right\|+\frac{1}{4}\left\|A\right\|^2+\frac{1}{2}\|A\|\left\|A^2\right\|^{\frac{1}{2}}\\
&= \left(\frac{1}{2}\|A\|+\frac{1}{2}\left\|A^2\right\|^{\frac{1}{2}}\right)^2.
\end{aligned}$$

Thus, the inequality in Corollary 2.6 also improves on (1.3).

Next we obtain the following inequality for the numerical radius of the sum of n operators which generalizes Theorem 2.5.

Theorem 2.8. *Let $A_i \in \mathcal{B}(\mathbb{H}),\ i=1,2,\ldots,n$. Then*

$$w^{2r}\left(\sum_{i=1}^n A_i\right) \leq \frac{n^{2r-1}}{4}\left\|\sum_{i=1}^n \left(|A_i|^{2r}+|A_i^*|^{2r}\right)\right\|+\frac{n^{2r-1}}{2}\left(\sum_{i=1}^n w\left(|A_i|^r|A_i^*|^r\right)\right),$$

for all $r \geq 1$.

Proof. Let $x \in \mathbb{H}$ with $\|x\|=1$. Then from Lemma 2.4 we get,

$$\begin{aligned}
\left|\left\langle\left(\sum_{i=1}^n A_i\right)x,x\right\rangle\right|^{2r} &= \left|\sum_{i=1}^n \langle A_i x,x\rangle\right|^{2r}\\
&\leq \left(\sum_{i=1}^n |\langle A_i x,x\rangle|\right)^{2r}\\
&\leq n^{2r-1}\left(\sum_{i=1}^n |\langle A_i x,x\rangle|^{2r}\right).
\end{aligned}$$

Proceeding similarly as in the proof of Theorem 2.5 we get the required inequality. □

Our next result reads as follows:

Theorem 2.9. *Let $A, B \in \mathcal{B}(\mathbb{H})$ be selfadjoint. Then*

$$\|A+B\| \leq \sqrt{w^2(A+\mathrm{i}B) + \|A\|\|B\| + w(BA)} \leq \|A\| + \|B\|.$$

Proof. Let $x \in \mathbb{H}$ be such that $\|x\| = 1$. Then we have,

$$\begin{aligned}
\|A+B\|^2 &= w^2(A+B) \\
&= \sup_{\|x\|=1} |\langle (A+B)x, x\rangle|^2 \\
&\leq \sup_{\|x\|=1} \left(|\langle Ax, x\rangle| + |\langle Bx, x\rangle|\right)^2 \\
&= \sup_{\|x\|=1} \left(|\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 + 2|\langle Ax, x\rangle||\langle Bx, x\rangle|\right) \\
&= \sup_{\|x\|=1} \left(|\langle Ax, x\rangle + \mathrm{i}\langle Bx, x\rangle|^2 + 2|\langle Ax, x\rangle\langle x, Bx\rangle|\right) \\
&\leq \sup_{\|x\|=1} \left(|\langle (A+\mathrm{i}B)x, x\rangle|^2 + \|Ax\|\|Bx\| + |\langle Ax, Bx\rangle|\right), \text{by Lemma 2.3} \\
&= \sup_{\|x\|=1} \left(|\langle (A+\mathrm{i}B)x, x\rangle|^2 + \|Ax\|\|Bx\| + |\langle BAx, x\rangle|\right)^2 \\
&\leq w^2(A+\mathrm{i}B) + \|A\|\|B\| + w(BA).
\end{aligned}$$

Hence,

$$\|A+B\| \leq \sqrt{w^2(A+\mathrm{i}B) + \|A\|\|B\| + w(BA)}.$$

It is easy to verify that $w^2(A+\mathrm{i}B) \leq \|A^2+B^2\|$. Therefore, we have

$$w^2(A+\mathrm{i}B) + \|A\|\|B\| + w(BA) \leq (\|A\| + \|B\|)^2.$$

This completes the proof. □

Remark 2.10. We would like to remark that Theorem 2.9 gives better bound than the bound obtained by Moradi and Sababheh [11, Th. 2.4], namely, if $A, B \in \mathcal{B}(\mathbb{H})$ are selfadjoint then

$$\|A+B\| \leq \sqrt{w^2(A+\mathrm{i}B) + 2\|A\|\|B\|} \leq \|A\| + \|B\|.$$

Next we prove the following inequality.

Theorem 2.11. *Let $A_i, B_i, X_i \in \mathcal{B}(\mathbb{H}), i = 1, 2, \ldots, n$. Let f and g be two nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$w^r\left(\sum_{i=1}^n A_i^*X_iB_i\right) \leq \frac{n^{r-1}}{\sqrt{2}}w\left(\sum_{i=1}^n\left(\left[B_i^*f^2(|X_i|)B_i\right]^r + \mathrm{i}\left[A_i^*g^2(|X_i^*|)A_i\right]^r\right)\right),$$

for all $r \geq 1$.

Proof. Let $x \in \mathbb{H}$ with $\|x\| = 1$. Then we have,

$$
\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^n A_i^*X_iB_i\right)x, x\right\rangle\right|^r \\
&= \left|\sum_{i=1}^n \langle A_i^*X_iB_ix, x\rangle\right|^r \\
&\leq \left(\sum_{i=1}^n |\langle A_i^*X_iB_ix, x\rangle|\right)^r \\
&\leq n^{r-1}\left(\sum_{i=1}^n |\langle A_i^*X_iB_ix, x\rangle|^r\right), \quad \text{by Lemma 2.4} \\
&= n^{r-1}\left(\sum_{i=1}^n |\langle X_iB_ix, A_ix\rangle|^r\right) \\
&\leq n^{r-1}\left(\sum_{i=1}^n \|f(|X_i|)B_ix\|^r\,\|g(|X_i^*|)A_ix\|^r\right), \quad \text{by Lemma 2.1} \\
&= n^{r-1}\left(\sum_{i=1}^n \left\langle f^2(|X_i|)B_ix, B_ix\right\rangle^{\frac{r}{2}}\left\langle g^2(|X_i^*|)A_ix, A_ix\right\rangle^{\frac{r}{2}}\right) \\
&= n^{r-1}\left(\sum_{i=1}^n \left\langle B_i^*f^2(|X_i|)B_ix, x\right\rangle^{\frac{r}{2}}\left\langle A_i^*g^2(|X_i^*|)A_ix, x\right\rangle^{\frac{r}{2}}\right) \\
&\leq n^{r-1}\left(\sum_{i=1}^n \left\langle\left[B_i^*f^2(|X_i|)B_i\right]^r x, x\right\rangle^{\frac{1}{2}}\left\langle\left[A_i^*g^2(|X_i^*|)A_i\right]^r x, x\right\rangle^{\frac{1}{2}}\right), \quad \text{by Lemma 2.2} \\
&\leq \frac{n^{r-1}}{2}\left(\sum_{i=1}^n\left(\left\langle\left[B_i^*f^2(|X_i|)B_i\right]^r x, x\right\rangle + \left\langle\left[A_i^*g^2(|X_i^*|)A_i\right]^r x, x\right\rangle\right)\right) \\
&\leq \frac{n^{r-1}}{\sqrt{2}}\left(\left|\sum_{i=1}^n\left\langle\left[B_i^*f^2(|X_i|)B_i\right]^r x, x\right\rangle + \mathrm{i}\sum_{i=1}^n\left\langle\left[A_i^*g^2(|X_i^*|)A_i\right]^r x, x\right\rangle\right|\right), \\
&\qquad\qquad\qquad \text{as } |a+b| \leq \sqrt{2}|a+\mathrm{i}b|, \quad \forall a, b \in \mathbb{R} \\
&= \frac{n^{r-1}}{\sqrt{2}}\left|\left\langle\left(\sum_{i=1}^n\left(\left[B_i^*f^2(|X_i|)B_i\right]^r + \mathrm{i}\left[A_i^*g^2(|X_i^*|)A_i\right]^r\right)\right)x, x\right\rangle\right|
\end{aligned}
$$

$$\leq \frac{n^{r-1}}{\sqrt{2}} w \left(\sum_{i=1}^{n} \left(\left[B_i^* f^2(|X_i|) B_i \right]^r + \mathrm{i} \left[A_i^* g^2(|X_i^*|) A_i \right]^r \right) \right).$$

Therefore, taking supremum over $\|x\| = 1$, we get

$$w^r \left(\sum_{i=1}^{n} A_i^* X_i B_i \right) \leq \frac{n^{r-1}}{\sqrt{2}} w \left(\sum_{i=1}^{n} \left(\left[B_i^* f^2(|X_i|) B_i \right]^r + \mathrm{i} \left[A_i^* g^2(|X_i^*|) A_i \right]^r \right) \right). \qquad \square$$

Remark 2.12. Note that Theorem 2.11 indeed does not depend on the number n of summands in the case $r = 1$.

Considering $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, $0 \leq \alpha \leq 1$ in Theorem 2.11 we get the following corollary.

Corollary 2.13. *For $i = 1, 2, \ldots, n$, let $A_i, B_i, X_i \in \mathcal{B}(\mathbb{H})$. Then*

$$w^r \left(\sum_{i=1}^{n} A_i^* X_i B_i \right) \leq \frac{n^{r-1}}{\sqrt{2}} w \left(\sum_{i=1}^{n} \left(\left[B_i^* |X_i|^{2\alpha} B_i \right]^r + \mathrm{i} \left[A_i^* |X_i^*|^{2(1-\alpha)} A_i \right]^r \right) \right),$$

for all $r \geq 1$.

The following corollary is an easy consequence of Theorem 2.11.

Corollary 2.14. *For $i = 1, 2, \ldots, n$, let $X_i \in \mathcal{B}(\mathbb{H})$. Let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$w^r \left(\sum_{i=1}^{n} X_i \right) \leq \frac{n^{r-1}}{\sqrt{2}} w \left(\sum_{i=1}^{n} \left(f^{2r}(|X_i|) + \mathrm{i}\; g^{2r}(|X_i^*|) \right) \right),$$

for all $r \geq 1$.

In particular, taking $n = 1$, $r = 1$ and $f(t) = g(t) = t^{\frac{1}{2}}$ in Corollary 2.14 we get the following inequality which refines the second inequality in (1.2).

Corollary 2.15. *Let $A \in \mathcal{B}(\mathbb{H})$. Then*

$$w(A) \leq \frac{1}{\sqrt{2}} w(|A| + \mathrm{i}|A^*|).$$

Remark 2.16. It is easy to observe that $w^2(|A| + \mathrm{i}|A^*|) \leq \left\| |A|^2 + |A^*|^2 \right\|$. Therefore,

$$w^2(A) \leq \frac{1}{2} w^2(|A| + \mathrm{i}|A^*|) \leq \frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\|.$$

Hence, Corollary 2.15 is sharper than that in (1.2).

We note that Corollary 2.15 is also obtained in [11, Cor. 2.1] by Moradi and Sababheh. However, our approach is different here.

Next, we obtain an inequality which follows from Corollary 2.13.

Corollary 2.17. *Let $A, B \in \mathcal{B}(\mathbb{H})$. Then*

$$w^r(A^*B) \leq \frac{1}{2} w^2\left(|B|^r + \mathrm{i}|A|^r\right),$$

for all $r \geq 2$.

Remark 2.18. In [7], Dragomir proved that if $A, B \in \mathcal{B}(\mathbb{H})$ and $r \geq 1$ then

$$w^r(A^*B) \leq \frac{1}{2}\left\||B|^{2r} + |A|^{2r}\right\|.$$

For $r \geq 2$, from Corollary 2.17 we get,

$$w^r(A^*B) \leq \frac{1}{2} w^2\left(|B|^r + \mathrm{i}|A|^r\right) \leq \frac{1}{2}\left\||B|^{2r} + |A|^{2r}\right\|.$$

We would like to remark that for $r \geq 2$, the inequality in Corollary 2.17 is stronger than the Dragomir's inequality [7].

Finally, we obtain the following estimation.

Theorem 2.19. *Let $A_i, B_i, X_{ij} \in \mathcal{B}(\mathbb{H})$ for $i, j = 1, 2, \ldots n$. Then*

$$w\left(\sum_{i,j=1}^{n} A_j^* X_{ij} B_i\right) \leq \frac{1}{2}\|\mathcal{X}\| \left\|\sum_{i=1}^{n}(A_i^*A_i + B_i^*B_i)\right\|,$$

where

$$\mathcal{X} = \begin{pmatrix} X_{11} & X_{21} & . & . & . & X_{n1} \\ X_{12} & X_{22} & . & . & . & X_{n2} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ X_{1n} & X_{2n} & . & . & . & X_{nn} \end{pmatrix} \in \mathcal{B}\left(\sum_{i=1}^{n} \oplus \mathbb{H}\right).$$

Proof. Let $A = \begin{pmatrix} A_1 & 0 & . & . & . & 0 \\ A_2 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ A_n & 0 & . & . & . & 0 \end{pmatrix}$, $B = \begin{pmatrix} B_1 & 0 & . & . & . & 0 \\ B_2 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ B_n & 0 & . & . & . & 0 \end{pmatrix} \in \mathcal{B}\left(\sum_{i=1}^n \oplus \mathbb{H}\right)$.

Then, $A^*\mathcal{X}B = \begin{pmatrix} \sum_{i,j=1}^n A_j^* X_{ij} B_i & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & . & . & . & 0 \end{pmatrix}$ and so we have $w\left(\sum_{i,j=1}^n A_j^* X_{ij} B_i\right) = w(A^*\mathcal{X}B)$. Now, by Cauchy-Schwarz inequality we get,

$$\begin{aligned} w(A^*\mathcal{X}B) &= \sup_{x\in S_{\mathbb{H}}} |\langle A^*\mathcal{X}Bx, x\rangle| = \sup_{x\in S_{\mathbb{H}}} |\langle \mathcal{X}Bx, Ax\rangle| \\ &\leq \sup_{x\in S_{\mathbb{H}}} \|\mathcal{X}Bx\|\|Ax\| \leq \sup_{x\in S_{\mathbb{H}}} \|\mathcal{X}\|\|Bx\|\|Ax\| \\ &\leq \sup_{x\in S_{\mathbb{H}}} \frac{1}{2}\|\mathcal{X}\|(\|Bx\|^2 + \|Ax\|^2) = \sup_{x\in S_{\mathbb{H}}} \frac{1}{2}\|\mathcal{X}\|\langle (B^*B + A^*A)x, x\rangle \\ &= \frac{1}{2}\|\mathcal{X}\|\|A^*A + B^*B\| = \frac{1}{2}\|\mathcal{X}\| \left\| \sum_{i=1}^n (A_i^*A_i + B_i^*B_i) \right\|. \end{aligned}$$

Thus, we have the desired inequality. □

Remark 2.20. We note that the expression $\sum_{i,j=1}^n A_j^* X_{ij} B_i$ can also be written as $\sum_{i=1}^{n^2} C_i^* X_i D_i$ where $C_i \in \{A_j : 1 \leq j \leq n\}, X_i \in \{X_{ij} : 1 \leq i, j \leq n\}, D_i \in \{B_j : 1 \leq j \leq n\}$ for all $i = 1, 2, \ldots n^2$. So, one can estimate $w\left(\sum_{i,j=1}^n A_j^* X_{ij} B_i\right)$ as in Theorem 2.11.

Declaration of competing interest

None declared.

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